SOUL THEOREM FOR 4-DIMENSIONAL TOPOLOGICALLY REGULAR OPEN NONNEGATIVELY CURVED ALEXANDROV SPACES

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Abstract. In this paper, we study the topology of topologically regular 4-dimensional open non-negatively curved Alexandrov spaces. These spaces occur naturally as the blow-up limits of compact Riemannian manifolds with lower curvature bound. These manifolds have also been studied by Yamaguchi in his preprint [Yam02]. Our main tools are gradient flows of semi-concave functions and critical point theory for distance functions, which have been used to study the 3-dimensional collapsing theory in the paper [CaoG10]. The results of this paper will be used in our future studies of collapsing 4-manifolds, which will be discussed elsewhere.

0. Introduction

The topology of noncompact manifold with a complete metric of nonnegative sectional curvatures was studied by Gromoll-Meyer in [GM69] and Cheeger-Gromoll in [CG72].

Theorem 0.1 (Soul Theorem, [CG72])**.** *Let M^n be an n-dimensional noncompact manifold with a complete metric of nonnegative sectional curvature. Then there exists a compact totally geodesic embedded submanifold $S \subset M$ with nonnegative sectional curvature, such that M^n is diffeomorphic to the normal bundle $\nu(S)$ of S in M^n.*

If in addition to the assumption above, there exists $p \in M^n$ such that the sectional curvatures at p are all positive, then M^n is diffeomorphic to $\mathbb{R}^n$. This is called Cheeger-Gromoll soul conjecture. It was proved by Perelman in [Per94].

For Alexandrov spaces, (see [BGP92] and [BBI01] for basics of Alexandrov spaces) Perelman proved a similar result:

Theorem 0.2 (Soul Theorem for Alexandrov space, [Per91])**.** *Let X^n be an n-dimensional non-compact Alexandrov space with nonnegative curvature, then there exits a closed totally convex subset $S \subset X^n$, such that S is a deformation retraction of X^n.*

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Similar to the soul conjecture, Cao-Dai-Mei [CDM07, CDM09] proved that if in addition to the conditions of Theorem 0.2, one assumes that X^n has positive curvature in a metric ball, then X^n is contractible. Unlike the manifold case, Theorem 0.2 is the best topological result one can expect, in the sense that in general X^n is not homeomorphic to the normal bundle of S.

By Perelman's stability theorem, if Alexandrov space is the limit of sequence of Riemannian manifolds with lower curvature bound, then it's a topological manifold. In fact Kapovitch showed in [Kap02]

Theorem 0.3 ([Kap02])**.** *If X^n is the limit of a sequence of n-dimensional Riemannian manifold with the same lower curvature bound k, then $\Sigma_p X^n$ is homeomorphic to $(n-1)$-sphere $\mathbf{S}^{n-1}$ for any $p \in X^n$. Moreover all the iterated space of directions are homeomorphic to spheres.*

It's still unknown whether an Alexandrov space, which satisfies the conclusion of Theorem 0.3, can be realized as a limit of Riemannian manifolds with the same dimension and same lower curvature bound. In this paper, we consider the class of 4-dimensional topologically regular open nonnegatively curved Alexandrov spaces, in the sense that all space of directions are spheres. These Alexandrov spaces occurs naturally as the blow-up limits of compact Riemannian manifolds with lower curvature bound, thus play an important role in the study of collapsing under a lower curvature bound. We will prove the following

Theorem 0.4 (Main Theorem)**.** *Let X^4 be as above, S be a soul of X^4, then X^4 is homeomorphic a open disk bundle over S:*

(1) *If $\dim S = 0$, then X^4 is homeomorphic to $\mathbb{R}^4$;*
(2) *If $\dim S = 1$, then X^4 is homeomorphic to $\mathbb{R}^3$ bundle over $\mathbf{S}^1$;*
(3) *If $\dim S = 2$, then X^4 is homeomorphic to $\mathbb{R}^2$ bundle over S, where $S = \mathbf{S}^2, \mathbb{RP}^2, \mathbf{T}^2$ or $\mathbf{K}^2$.*
(4) *If $\dim S = 3$, then X^4 is homeomorphic to line bundle over S, where $S = \mathbf{S}^3/\Gamma, \mathbf{T}^3/\Gamma, (\mathbf{S}^2 \times \mathbf{S}^1)/\Gamma$, and Γ some subgroup of isometric group of S acting freely on S.*

This theorem will be used to study the collapsing 4-manifolds, which will discussed elsewhere. Theorem 0.4 has also been studied in the preprint [Yam02] Chap 15,16 in a traditional way. Our main tools are the gradient flow of semi-concave functions and Perelman's version of Fibration theorem, which have been used extensively in [CaoG10] to study the 3-dimensional collapsing manifolds under a lower curvature bound.

1. Construction of Soul

In this section we recall Cheeger-Gromoll's construction of soul for X^n, where X^n is an n-dimensional non-negatively curved open complete Alexandrov space. (c.f. [CG72], [Per91]).

Fix $p \in X^n$, the Busemann function can be defined by

$$b(x) = \lim_{t \to \infty} [\mathrm{d}(x, \partial(B(p,t))) - t]. \tag{1.1}$$

where $B(p,t)$ is the ball centered at p with radius t. Denote the super-level set $b^{-1}([a, +\infty))$ by Ω^a. We have

Proposition 1.1 ([CG72], [Per91], [Wu79])**.** *Let X^n, Ω^a be as above, then the following hold*

(1) *The Busemann function b is concave and bounded above.*
(2) *Ω^a is compact and totally convex for all $a \le a_0 := \max_{x \in X^n} b(x)$.*
(3) *$a < b \le a_0$ implies $\Omega^b \subset \Omega^a$ and*

$$\Omega^b = \{x \in \Omega^a |\, \mathrm{d}(x, \partial\Omega^a) \le b - a\}$$

(4) *There is a filtration of $\Omega(0) := \Omega^{a_0}$ by totally convex sets:*

$$\Omega(0) \supset \Omega(1) \supset \cdots \supset \Omega(k)$$

where $\Omega(i+1)$ is the maximum set of $f_i(x) = \mathrm{d}_{\Omega(i)}(x, \partial\Omega(i))$, and $\partial\Omega(k) = \varnothing$.

We call $S = \Omega(k)$ a soul of the type (s, m), if the dimension of the soul is s and the dimension of $\Omega(0)$ is m.

We call a geodesic $\gamma : (-\infty, +\infty) \to X^n$ a line in a metric space X^n, if $\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|$ for $\forall t, s \in \mathbb{R}$. The splitting theorem reduces our discussion of 4-dimension to 3-dimension when X^4 admits a line.

Theorem 1.2 ([GP89])**.** *Let X^n be open non-negatively curved Alexandrov space and assume that X^n admits a line, then X^n splits isometrically as $X^n = N^{n-1} \times \mathbb{R}$, where N^{n-1} is a $(n-1)$ dimensional open non-negatively curved Alexandrov space.*

In order to handle the non-smooth metric Alexandrov space, Perelman's Stability Theorem and his version of Fibration Theorem are extensively used in this paper. Let's recall these results.

Theorem 1.3 (Stability Theorem [Per91], [Kap07])**.** *Let $\{X^n_\alpha\}_{\alpha=1}^\infty$ be a sequence of n-dimensional Alexandrov spaces with* $\operatorname{curv} \ge -1$ *converging to an Alexandrov space with same dimension: $\lim_{\alpha \to \infty} X^n_\alpha = X^n$. Then X^n_α is homeomorphic to X^n for α large.*

The stability theorem for pointed spaces can be stated in a similarly way. The fibration theorem states that

Theorem 1.4 (Fibration Theorem [Per91, Per93]). *Let X^n be an n-dimensional Alexandrov space, U a domain in X^n, $f : U \to \mathbb{R}^k$ be an admissible function, having no critical point and proper on U, then it's restriction to this domain is a locally trivial fiber bundle.*

We refer to [Per91] for the definitions of admissible functions and regular map.

2. Soul theorem for 3-dimensional Alexandrov space

The topology of 3-dimensional open non-negatively curved Alexandrov space was studied in [SY00], (cf also [CaoG10]).

Theorem 2.1 ([SY00]). *Let X^3 be an open complete 3-manifold with a possibly singular metric of non-negative curvature. Suppose that X^3 is oriented and S^s is a soul of X^3. Then the following is true.*

(1) *When $\dim(S^s) = 1$, then the soul of X^3 is isometric to a circle. Moreover, its universal cover $\tilde{X}^3$ is isometric to $\tilde{X}^2 \times \mathbb{R}$, where $\tilde{X}^2$ is homeomorphic to $\mathbb{R}^2$;*
(2) *When $\dim(S^s) = 2$, then the soul of X^3 is homeomorphic to $\mathbf{S}^2/\Gamma$ or $\mathbf{T}^2/\Gamma$. Furthermore, X^3 is isometric to one of four spaces: $\mathbf{S}^2 \times \mathbb{R}$, $\mathbb{RP}^2 \ltimes \mathbb{R} = (\mathbf{S}^2 \times \mathbb{R})/\mathbb{Z}_2$, $\mathbf{T}^2 \times \mathbb{R}$ or $\mathbf{K}^2 \ltimes \mathbb{R} = (\mathbf{T}^2 \times \mathbb{R})/\mathbb{Z}_2$ and $\mathbb{RP}^2 \ltimes \mathbb{R}$ which is homeomorphic to $[\mathbb{RP}^3 - D^3]$;*
(3) *When $\dim(S^s) = 0$, then the soul of X^3 is a single point and X^3 is homeomorphic to $\mathbb{R}^3$.*

Throughout this paper $\mathbf{S}^n$ denotes the standard n-sphere, $\mathbf{T}^n$ denotes the n-dimensional torus, $\mathbf{K}^2$ denotes the Klein bottle and D^3 denotes the standard 3-ball.

3. Proof of the main theorem

A key observation by K. Grove is that the distance function to the soul has no critical point in $X^n - S^s$ when X^n is a smooth Riemannian manifold. However this is no longer true for Alexandrov spaces even for topologically regular one. For example let L be the closed half strip $\{(x, y) \in \mathbb{R}^2 | x \geq 0, 0 \leq y \leq 1\}$, then the double of L which denoted by $\mathrm{dbl}(L)$ is a 2-dimensional Alexandrov space with non-negative curvature, which is homeomorphic to $\mathbb{R}^2$ and has point soul $(0, \frac{1}{2})$, it's clear that the distance function has critical points $(0, 0)$ and $(0, 1)$. However for Alexandrov space, [CaoG10] derived a modified result similar to Grove's observation.

Proposition 3.1 ([CaoG10] Proposition 2.5)**.** *The function* $f(x) = \mathrm{d}_{X^n}(x, A)$ *has no critical point for* $x \in [X^n - \Omega(0)]$*, where* $A \subset \Omega(0)$*.*

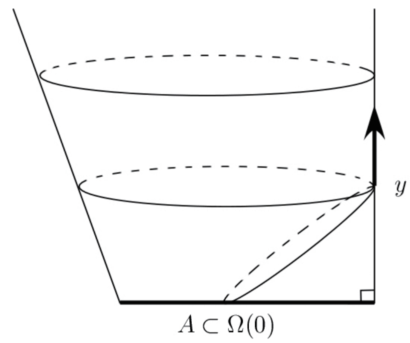

FIGURE 1. $\mathrm{d}_{X^n}(x, A)$ has no critical point in $[X^n - \Omega(0)]$

For simplicity we assume that $a_0 = 0$ by adding a constant if needed. Let $A = \Omega(0)$. Using Proposition 3.1, we see that $f(x) = \mathrm{d}(x, A)$ has no critical value within $(\varepsilon, +\infty)$. It follows by Perelman's Fibration Theorem, that $X^n \cong B_{X^n}(\Omega(0), \varepsilon)$ for $\varepsilon > 0$, where $B_X(A, r)$ denoted the set of points with distance $\leq r$ to set A in metric X.

We will divide the proof of our Main Theorem into the flowing cases.

3.1. **Soul of the type** $(s = 3, m = 3)$**,** $X^4 = S \times \mathbb{R}^1$ **or** $(S \times \mathbb{R}^1)/\mathbb{Z}_2$**.**

Proof. (cf. the proof of Theorem 2.21 in [CaoG10]) In this case, $S = \Omega(0)$ and has dimension 3. Since X^4 is topologically regular, S is a topological manifold and non-negatively curved, hence by Hamilton's classification of 3-dimensional manifolds with non-negatively Ricci curvature, S is homeomorphic to $\mathbf{S}^3/\Gamma$, $\mathbf{T}^3/\Gamma$ or $\mathbf{S}^2 \times \mathbf{S}^1/\Gamma$, where Γ is a subgroup of the isometry group of $\mathbf{S}^3, \mathbf{T}^3, \mathbf{S}^2 \times \mathbf{S}^1$.

For $p \in S$, we know $\Sigma_p(S)$ is homeomorphic to $\mathbf{S}^2$, which divide $\Sigma_p(X^4) \cong \mathbf{S}^3$ into two parts, denoted by $A^{\pm}$. Since S is totally convex, $\Sigma_p(S)$ is convex in $\Sigma_p(X^4)$, therefore $r_{\Sigma_p(S)}|_{A^{\pm}}$ have a unique maximum point $\xi^{\pm}$ in $A^{\pm}$. Denote the maximum values by $\ell^{\pm}$, i.e. $r_{\Sigma_p(S)}(\xi^{\pm}) = \ell^{\pm}$. Since $\Omega(0) = S$ is the set of maximum points for Busemann function, by the first variation theorem, we know $\ell^{\pm} \geq \pi/2$. On the other hand if $\gamma : [0, \ell] \to \Sigma_p(X^4)$ is a shortest geodesic connecting ξ^- to ξ^+, and let $t_0 \in [0, \ell]$ satisfying $\gamma(t_0) \in \Sigma_p(S)$. By triangle inequality we know

$$
\begin{aligned}
\mathrm{d}(\xi^-,\xi^+) &= \mathrm{d}(\xi^-,\gamma(t_0)) + \mathrm{d}(\gamma(t_0),\xi^+) \\
&\geq \mathrm{d}(\Sigma_p(S),\xi^-) + \mathrm{d}(\Sigma_p(S),\xi^+) \\
&= \ell^- + \ell^+ \\
&\geq \frac{\pi}{2} + \frac{\pi}{2} \\
&= \pi
\end{aligned} \tag{3.1}
$$

Note that $\mathrm{curv}(\Sigma_p(X^4)) \geq 1$ implies $\mathrm{diam}(\Sigma_p(X^4)) \leq \pi$, hence the inequalities in (3.1) are equalities, in particular $\ell^- = \ell^+ = \pi/2$ and $\Sigma_p(X^4)$ is the spherical suspension over $\Sigma_p(S)$, i.e. $T_p(X^4)$ splits isometrically as $T_p(S) \times \mathbb{R}^1$. Hence we have a normal line bundle over S. By passing to the double cover, we can assume that this line bundle is trivial, therefore S separates X^4 into two parts and X^4 has two ends. Now it's easy to see X^4 admits a line, by splitting theorem, X^4 is isometric to $S \times \mathbb{R}^1$. □

3.2. **Soul of the types** $(s = 0$ **or** $2, m = 3)$**,** $X^4 \cong \mathbb{R}^4$ **or** $\mathbb{R}^2 \hookrightarrow X^4 \to \Sigma^2$ **where** $\Sigma^2 \cong \mathbf{S}^2/\Gamma, \mathbf{T}^2/\Gamma$**.**

Proof. In this case, $\Omega(0) \cong D^3$ or I-bundle over $S \cong \mathbf{S}^2, \mathbb{RP}^2, \mathbf{T}^2$ or $\mathbf{K}^2$ by Theorem 2.1. By the proof of §3.1, the interior of $\Omega(0)$ admits a normal line bundle. Thus, we only have to show that $B_{X^4}(\Omega(0),\varepsilon) \cong B_{X^4}(\Omega^{-\delta}(0),\varepsilon)$ for $0 < \varepsilon \ll \delta$, where $\Omega^{-\delta}(0) := \{x \in \Omega | \, \mathrm{d}(x, \partial\Omega(0)) \geq \delta\}$.

By Proposition 3.1, $r_{\partial\Omega(0)}(x) = \mathrm{d}(\partial\Omega(0), x)$ has no critical point for $x \in X^4 \setminus \Omega(0)$. When restrict to $\Omega(0)$, $r_{\partial\Omega(0)}$ is concave, hence it has no critical value in $(0,a)$ for a small enough. Combine these two facts, we know there exits $\delta > 0$ such that $r_{\partial\Omega(0)}$ has no critical point in $B_{X^4}(\partial\Omega(0), 100\delta)$.

By the lower semi continuity of the norm of gradient of λ-concave function $\nabla r_{\partial\Omega(0)}$ (cf. [Petr07] Corollary 1.3.5), $r_{\partial\Omega(0)}$ has no critical point in $B_{X^4}(\Omega(0) \setminus \Omega^{-2\delta}(0),\varepsilon) - \partial\Omega(0)$ for $\varepsilon \ll \delta$ small enough. Therefore $r_{\partial\Omega(0)}$ is regular in $B_{X^4}(\Omega^{-\varepsilon}(0) \setminus \Omega^{-2\delta}(0),\varepsilon)$, by Fibration Theorem we have

$$
B_{X^4}(\partial\Omega(0),\varepsilon) \cong B_{X^4}(\Omega(0) \setminus \Omega^{-2\delta}(0),\varepsilon) \tag{3.2}
$$

for $\varepsilon \ll \delta$, see Figure 2.

By the proof of §3.3, we know $B_{X^4}(\partial\Omega(0),\varepsilon)$ is homeomorphic to a D^2 bundle over $\partial\Omega(0)$

$$
D^2 \hookrightarrow B_{X^4}(\partial\Omega(0),\varepsilon) \to \partial\Omega(0) \tag{3.3}
$$

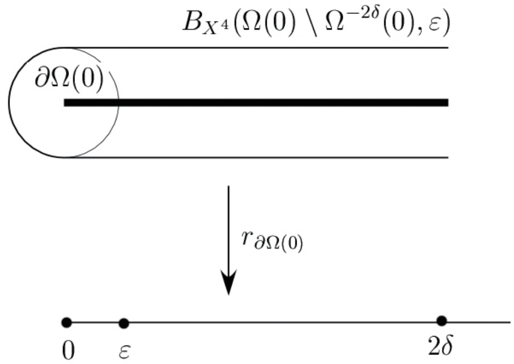

FIGURE 2. Bundle structure around $\partial\Omega(0)$

since $\partial\Omega^{-\varepsilon/2}(0) \cong \partial\Omega(0)$ and there is a normal line bundle over $\partial\Omega^{-\varepsilon/2}(0)$, by passing to the double cover one can assume this line bundle is trivial, let Γ be the $(\varepsilon/100)$-section of this line bundle. Clearly $\Gamma \subset B_{X^4}(\partial\Omega(0), \varepsilon)$, which implies that the bundle (3.3) admits a global nowhere vanishing section, hence it is a trivial D^2 bundle. By the homeomorphism (3.2), we have a trivial D^2 bundle

$$B_{X^4}(\Omega(0) \setminus \Omega^{-2\delta}(0), \varepsilon) \cong \partial\Omega(0) \times D^2 \tag{3.4}$$

Now consider the function $r_{\Omega^{-10\delta}}(x)$, which has no critical point in $B_{X^4}(\Omega(0) \setminus \Omega^{-9\delta}(0), \varepsilon) - \partial\Omega(0)$, hence we have a gradient flow of $r_{\Omega^{-10\delta}(0)}$ on

$$\begin{aligned} B_{X^4}(\Omega^{-\delta}(0) \setminus \Omega^{-2\delta}(0), \varepsilon) &\cong (-\varepsilon, +\varepsilon) \times \partial\Omega(0) \times (\delta, 2\delta) \\ &\cong \partial\Omega(0) \times D^2 \end{aligned} \tag{3.5}$$

for $\varepsilon \ll \delta$ small enough, where the homeomorphism follows from the facts that $r_{\partial\Omega(0)}$ is concave in the interior of $\Omega(0)$ and that there is a normal line bundle over the interior of $\Omega(0)$, see §3.1.

Since the bundle (3.4) and it's sub-bundle (3.5) are both trivial bundles, one can extend the gradient flow of $r_{\Omega^{-10\delta}(0)}$ on $B_{X^4}(\Omega^{-\delta}(0) \setminus \Omega^{-2\delta}(0), \varepsilon)$ to a gradient-like flow on $B_{X^4}(\Omega(0) \setminus \Omega^{-2\delta}(0), \varepsilon)$, which give the homeomorphism (see Figure 3):

$$B_{X^4}(\Omega(0), \varepsilon) \cong B_{X^4}(\Omega^{-2\delta}(0), \varepsilon)$$

It follows from the proof of §3.1 that $B_{X^4}(\Omega^{-2\delta}(0), \varepsilon)$ is homeomorphic to the normal line bundle over $\Omega^{-2\delta}(0)$, hence combine with Theorem 2.1, our main theorem holds in these two cases.

$\square$

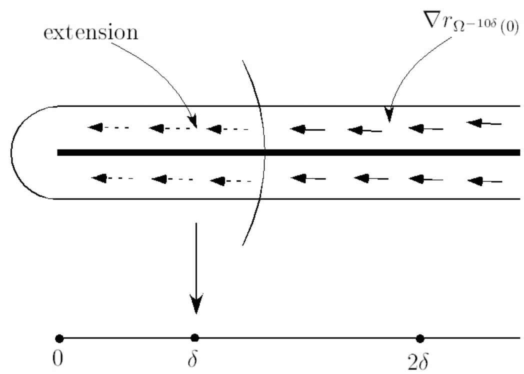

FIGURE 3. Gradient-like flow on $B_{X^4}(\partial\Omega(0), \varepsilon)$

3.3. **Soul of the type** $(s = 2, m = 2)$**,** $\mathbb{R}^2 \hookrightarrow X^4 \to S$**,** $S = \mathbf{S}^2/\Gamma, \mathbf{T}^2/\Gamma$**.**

Proof. In this case, S is 2-dimensional surface with non-negatively curvature. Thus S is homeomorphic to $\mathbf{S}^2, \mathbf{T}^2, \mathbb{RP}^2$ or $\mathbf{K}^2$, by Splitting Theorem, if $S = \mathbf{T}^2$ or $S = \mathbf{K}^2$ the universal cover $\widetilde{X^4}$ splits isometrically as $N^2 \times \mathbb{R}^2$, thus the theorem follows from the fact that N^2 is homeomorphic to $\mathbb{R}^2$. Now we consider the cases where $S = \mathbf{S}^2$ or $\mathbb{RP}^2$.

Let $\{p_i\}_{i=1}^N$ be the set of extremal points on S, where the extremal points in Alexandrov surface is defined to be the points satisfying $\operatorname{diam}(\Sigma_{p_i}(S)) \leq \pi/2$. Note that $\Sigma_{p_i}(S) = \mathbf{S}^1$ is a convex subset of $\Sigma_{p_i}(X^4) = \mathbf{S}^3$, hence for $\varepsilon > 0$, we have $B_{T_{p_i}(X^4)}(T_{p_i}(S), \varepsilon) = T_{p_i}(S) \times D^2$. Then by Stability Theorem, we know there exits $\delta > 0$ such that $B_{X^4}(p_i, \delta) \cap B_{X^4}(S, \varepsilon) \cong D^4$ and for $\varepsilon \ll \delta$, we have disc bundle

$$D^2 \hookrightarrow B_{X^4}(p_i, \delta) \cap B_{X^4}(S, \varepsilon) \xrightarrow{\pi_i} B_S(p_i, \delta). \tag{3.6}$$

where π_i is the bundle projection map. In particular $\pi_i^{-1}(\partial B_S(p_i, \delta)) = D^2 \times S^1$

For $p \in S \setminus (\cup_{i=1}^N B_S(p_i, \delta/10))$, by our assumption $\Sigma_p(S) > \pi/2$, thus there exits $\delta' > 0$ and a admissible map $F_p = (f_1, f_2) : B_S(p, \delta') \to \mathbb{R}^2$ which is regular $B_S(p, \delta')$, by the lower semi-continuity of gradient of semi-concave functions, we know F_p is regular in $B_{X^4}(p, \delta'')$, for some $\delta'' > 0$ satisfying $\delta'' \leq \delta \ll \varepsilon$. Thus we have a fiber bundle:

$$N \hookrightarrow B_{X^4}(p, \delta'') \xrightarrow{\pi} D^2 \cong B_S(p, \delta'') \tag{3.7}$$

Let $f(x) = \mathrm{d}(x, S)$. Since $G_p = (f_1, f_2, f)$ is regular in the domain $A_{X^4}(S, \varepsilon/100, \varepsilon) \cap B_{X^4}(p, \delta'')$, for $\varepsilon \ll \delta$, where $A_X(S, \varepsilon_1, \varepsilon_2)$ denoted

the annular region, i.e. all points have distance to S between ε_1 and ε_2. It follows from the Fibration Theorem that we have a fiber bundle:

$$\mathbf{S}^1 \hookrightarrow A_{X^4}(S, \varepsilon/100, \varepsilon) \cap B_{X^4}(p, \delta'') \to D^2 \times I$$

where I is a open interval. Thus $\partial N = \mathbf{S}^1$, by generalized Margulis Lemma (cf. [FY92]), $N \cong D^2$.

Now we can glue the D^2 bundle together, (this part is similar to Yamaguchi's construction in [Yam02] Page102). Let $S(\frac{\delta}{2}) = S - \cup_{i=1}^N B(p_i, \frac{\delta}{2})$. By construction we have a D^2 bundle over $S(\frac{\delta}{2})$:

$$D^2 \hookrightarrow B_{X^4}(S(\frac{\delta}{2}), \varepsilon) \xrightarrow{\pi} S(\frac{\delta}{2}) \tag{3.8}$$

Hence $\pi^{-1}(\partial B_S(p_i, \frac{\delta}{2})) = D^2 \times S^1$. Consider the gradient flow of $r_{p_i}(\cdot) = \mathrm{d}(p_i, \cdot)$ on $A_S(p_i, \delta/2, \delta)$, again by the lower semi-continuity of $|\nabla r_{p_i}|$, for $\varepsilon \ll \delta$ small enough, r_{p_i} is regular in $B_{X^4}(A_S(p_i, \delta/2, \delta), \varepsilon)$ hence provide a homeomorphism ϕ between $F_i := \pi^{-1}(\partial B_S(p_i, \frac{\delta}{2}))$ and $G_i := \pi^{-1}(\partial B_S(p_i, \delta))$. Clearly $\partial B_S(p_i, \delta/2)$ isotopic to $\partial B_S(p_i, \delta)$ in $B_S(p_i, 2\delta)$, hence when restricted to the boundaries $\partial F_i = \mathbf{S}^1 \times \mathbf{S}^1$ and $\partial G_i = \mathbf{S}^1 \times \mathbf{S}^1$, ϕ is isotopic to the identity, therefore we can glue the D^2 bundle together. □

3.4. **Soul of the types** $(s = 1, m = 1, 2, 3)$, $\mathbb{R}^3 \hookrightarrow X^4 \to \mathbf{S}^1$.

Proof. If the soul is $\mathbf{S}^1$, then the universal cover $\widetilde{X^4}$ admits a line by the totally convexity of soul, so $\widetilde{X^4} = N^3 \times \mathbb{R}$, where N^3 is homeomorphic to $\mathbb{R}^3$ by Theorem 2.1, thus Theorem 0.4 holds in this case. □

3.5. **Soul of the type** $(s = 0, m = 2)$, $X^4 \cong \mathbb{R}^4$.

Proof. Since the normal bundle over point soul S is homeomorphic to $\mathbb{R}^4$, it's enough to show that $B_{X^4}(\Omega(0), \varepsilon) \cong D^4$. It's clear that $\Omega(0) \cong D^2$ since the soul is a point. By the proof of §3.3 and the fact that D^2 is contractible, we see that all D^2 bundle over $\Omega(0) \cong D^2$ is trivial, we have

$$B_{X^4}(\Omega^{-100\varepsilon}, \varepsilon) \cong D^2 \times D^2 \cong D^4$$

We claim that $B_{X^4}(\partial\Omega(0), \varepsilon) \cong \mathbf{S}^1 \times D^3$. Assume the claim first, by the proof of §3.2, the gradient of $\mathrm{d}_{X^4}(\Omega^{-10\delta}(0), \cdot)$ can be extend to $B_{X^4}(\partial\Omega(0), \varepsilon)$, and will give the homeomorphism from $B_{X^4}(\Omega^{-100\varepsilon}, \varepsilon) \cong D^4$ to $B_{X^4}(\Omega(0), \varepsilon)$.

Proof of the Claim: Let $\{p_i\}_{i=1}^N$ be the set of extremal points in $\partial\Omega(0)$. By the Stability Theorem, $B_{X^4}(p_i, \varepsilon) \cong D^4$. Let γ_i be the boundary curve connection p_i to p_{i+1} with the understanding that $p_{N+1} = p_1$. One can assume γ_i is short enough such that d_{p_i} has no critical point

in $B_{X^4}(\gamma_i^{-\delta}, \varepsilon)$, where $\delta \gg \varepsilon$, $\gamma_i^{-\delta}$ is the sub-curve of γ_i defined by $\{x \in \gamma_i | \, \mathrm{d}(x, p_i) \geq \delta \ and \ \mathrm{d}(x, p_{i+1}) \geq \delta\}$. Hence by Fibration Theorem it's a locally trivial fiber bundle,

$$N^3 \hookrightarrow B_{X^4}(\gamma_i, \varepsilon) \to \gamma_i^{-\delta}$$

Since $B_{X^4}(p_i, \varepsilon) \cong D^4$, we have $S_{X^4}(p_i, \delta) \cap B_{X^4}(\gamma_i^{-\delta}, \varepsilon) \cong D^3$, hence $N^3 \cong D^3$. This finishes the proof. $\square$

3.6. Soul of the type $(s = 0, m = 1)$, $X^4 \cong \mathbb{R}^4$.

Proof. The proof is identically same as Subcase 3.1 of Theorem 2.21 in [CaoG10], we omit it here. $\square$

3.7. Soul of the type $(s = 0, m = 0)$, $X^4 \cong \mathbb{R}^4$.

Proof. Let $p = S$ be a soul and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of number such that $\lim_{n\to\infty} \alpha_n = \infty$. It's clear that $\lim_{n\to\infty}(\alpha_n X^4, p) = (T_p(X^4), O)$. Then it follows by Stability Theorem that $B_{\alpha_n X^4}(p, \varepsilon) \cong B_{T_p X^4}(O, \varepsilon) \cong \mathbb{R}^4$. Since $\mathrm{d}_p(x)$ has no critical point in $X^4 \setminus p$, we conclude that $X^4 \cong \mathbb{R}^4$ by Perelman's Fibration theorem. $\square$

This completes the proof of the Main Theorem.

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